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LETTER TO THE EDITOR

A nonunitary version of massless quantum electrodynamics possessing a critical point

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Abstract. Recently, it has been observed that a quantum field theory need not be hermitian to have a real, positive spectrum. What seems to be required is symmetry under combined parity and time-reversal transformations. This idea is extended to massless electrodynamics, in which the photon couples to the axial-vector current with an imaginary coupling constant. The eigenvalue condition necessary for the finiteness of the theory can now be solved; the value for the charge appears to be stable order-by-order. Similarly, the semiclassical Casimir model for the fine-structure constant yields a positive value.

Recently, there have been investigations of quantum theories whose Hamiltonians are nonhermitian. It has been found that the energy spectra are real and positive when these theories respect $\mathcal{PT}$ invariance, where $\mathcal{P}$ and $\mathcal{T}$ represent parity and time reversal. A class of quantum-mechanical theories having this property is defined by the Hamiltonian [1]

$$H = p^2 - (\mathrm{i}x)^N \qquad (N \text{ real}). \tag{1}$$

For all $N \geqslant 2$ the spectrum of H is discrete, real, and positive¶. Note that this theory does not respect parity invariance; thus for all N (including $N = 4$) the expectation value of x is nonvanishing [1]. This surprising result is a consequence of the boundary conditions+.

Quantum field theories having this property have also been studied. A generalization of equation (1) to scalar quantum field theory is described by the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\partial\phi)^2 + \tfrac{1}{2}m^2\phi^2 - g(\mathrm{i}\phi)^N \qquad (N \geqslant 2). \tag{2}$$

This theory is not symmetric under $\mathcal{P}$ or $\mathcal{T}$ separately, but it is invariant under the product $\mathcal{PT}$. The Hamiltonian for this theory is not hermitian and thus the theory is not unitary in the conventional sense. However, there is strong evidence that the energy spectrum is real and bounded below [4]. One can heuristically understand positivity in the context of the weak-coupling expansion for the case $N = 3$. The Lagrangian for this theory is

$$\mathcal{L} = \tfrac{1}{2}(\partial\phi)^2 + \tfrac{1}{2}m^2\phi^2 + g\mathrm{i}\phi^3. \tag{3}$$

In a conventional $g\phi^3$ theory the weak-coupling expansion is real, and (apart from a possible overall factor of g) the Green functions are formal power series in g^2. These series are not

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¶ When $N < 2$ some eigenvalues are real and others are complex because $\mathcal{PT}$ symmetry is spontaneously broken. See [2].

+ For a discussion of the effect of boundary conditions see [3].

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Borel summable because they do not alternate in sign. Nonsummability reflects the fact that the spectrum of the underlying theory is not bounded below. However, when we replace g by $\mathrm{i}g$, the perturbation series remains real but now alternates in sign. The perturbation series is now summable and this suggests that the underlying theory has a real positive spectrum.

We emphasize that replacing g by $\mathrm{i}g$ in a ϕ^3 field theory or g by $-g$ in a ϕ^4 field theory gives a nonhermitian Hamiltonian. However, the $\mathcal{PT}$ invariance of the resulting theory is crucial and appears to guarantee that the energy spectrum is positive.

The purpose of this letter is extend these notions to quantum electrodynamics†. In particular, we wish to discuss the case of massless quantum electrodynamics and to re-examine the program of Johnson *et al* [6]. In brief, the objective of their program is to find a critical value of the coupling constant e in the Lagrangian describing massless quantum electrodynamics

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \bar{\psi}\gamma^{\mu}\frac{1}{\mathrm{i}}\partial_{\mu}\psi + e\bar{\psi}\gamma^{\mu}A_{\mu}\psi. \tag{4}$$

The coupling constant e is determined by the condition that the theory be entirely finite. The mass shift in this theory is finite because the unrenormalized masses are zero. Thus, the only possible infinite quantities are associated with the three renormalization constants Z_1, Z_2, and Z_3. However, to any order in powers of e, it is possible to find a gauge in which $Z_1 = Z_2$ is finite. Thus, the only remaining divergent quantity is $1/Z_3$. Demanding that this be finite translates into an eigenvalue condition on the fine structure constant

$$\alpha = \frac{e^2}{4\pi}. \tag{5}$$

This eigenvalue condition takes the form

$$F_1(\alpha) = 0. \tag{6}$$

The function $F_1(\alpha)$ has been calculated to three loops (four terms) in weak-coupling perturbation theory:

$$F_1(\alpha) = \frac{4}{3}\left(\frac{\alpha}{4\pi}\right) + 4\left(\frac{\alpha}{4\pi}\right)^2 - 2\left(\frac{\alpha}{4\pi}\right)^3 - 46\left(\frac{\alpha}{4\pi}\right)^4 + \cdots. \tag{7}$$

The first two terms in this series were calculated by Jost and Luttinger [7]. Unfortunately, with just two terms the only nontrivial solution of equation (6) is negative, which gives an unphysical imaginary value for e. In a dramatic development, Rosner [8] calculated the third term in the series. The negative sign of his result is significant because now there is a positive root to the cubic polynomial equation obtained by truncating $F_1(\alpha)$ after three terms:

$$\alpha = 13.872. \tag{8}$$

Rosner's two-loop result is surprising because it is rational. His work suggests the conjecture that all of the coefficients in the expansion of $F_1(\alpha)$ might be rational, possibly reflecting a deep symmetry of the underlying massless theory [9]. This conjecture has recently gained support through the stunning calculation of the three-loop coefficient by Gorishny and coworkers [10–12]. The fourth-degree equation gives one positive nontrivial root for α:

$$\alpha = 3.969. \tag{9}$$

This value differs from the result in equation (8) by a factor of 3.5, which suggests that this nontrivial root is unstable.

One might wonder if a stable positive root can be found by first converting the expansion of $F_1(\alpha)$ to Padé form. The $(1, 1)$ Padé of the Rosner result gives no positive root at all. The

† This idea has been extended to quasi-exactly solvable potentials in quantum mechanics in Bender and Boettcher [5]. It has been extended to supersymmetric quantum field theory in Bender and Milton [5].

(1, 2) Padé of the four-term series gives one positive root, $\alpha = 0.814$, and the (2, 1) Padé gives $\alpha = 0.545$. There seems to be no sensible pattern to these numerical results.

The results regarding $\mathcal{PT}$-symmetric nonhermitian quantum field theories are intriguing because they suggest that it is possible to formulate a new kind of electrodynamics. Instead of coupling the A field to a vector current, why not couple this field to an axial-vector current? Of course, this coupling breaks parity symmetry. Therefore, we also replace e by ie, thereby breaking time-reversal invariance as well! The resulting $\mathcal{PT}$-symmetric, massless Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\psi\gamma^0\gamma^\mu\frac{1}{\mathrm{i}}\partial_\mu\psi + e\frac{1}{2}\psi\gamma^0\gamma^5\gamma^\mu A_\mu\psi. \tag{10}$$

We conjecture on the basis of our experience with scalar theories that the spectrum of this theory is physically acceptable in that it is bounded below.

Our conventions in equation (10) are as follows: γ^0 is antisymmetric and pure imaginary, $\gamma^0\gamma^\mu$ is symmetric and real, $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$ is antisymmetric and real, and $(\gamma^5)^2 = -1$. The fermion field ψ is expected to be complex, as are the operators x and p in equation (1) and ϕ in equations (2) and (3).

Like the conventional electrodynamics described by the Lagrangian (4), $\mathcal{L}$ in (10) possesses gauge invariance; $\mathcal{L}$ is invariant under the replacements

$$A^\mu \to A^\mu + \partial^\mu\Lambda \qquad \psi \to \mathrm{e}^{-\mathrm{i}e\gamma^5\Lambda}\psi. \tag{11}$$

Note that this gauge transformation on the fermion field is not a phase transformation when e is real; rather it changes the scale of ψ. However, the bilinear forms in the fermion field in the Lagrangian and in the energy-momentum tensor are all invariant.

Apart from a possible overall factor of $e\gamma^5$ in some of the Green functions, the Feynman rules for the Lagrangian (10) give precisely the same weak-coupling expansion as in the conventional massless quantum electrodynamics (4), except that α is now replaced by $-\alpha$. Thus, in this new and peculiar theory of quantum electrodynamics, the expansion of $F_1(\alpha)$ becomes

$$F_1(\alpha) = -\frac{4}{3}\left(\frac{\alpha}{4\pi}\right) + 4\left(\frac{\alpha}{4\pi}\right)^2 + 2\left(\frac{\alpha}{4\pi}\right)3 - 46\left(\frac{\alpha}{4\pi}\right)^4 + \cdots. \tag{12}$$

Now, we find that there *is* a nontrivial positive value α_2 satisfying the condition (6) when only the first two terms are retained:

$$\alpha_2 = 4.189. \tag{13}$$

If the first three terms are retained, the (unique) positive root is

$$\alpha_3 = 3.657 \tag{14}$$

which differs from α_2 by 12%. We feel that since α_2 is determined in effect from a (2, 0) Padé it is more reasonable to convert the three-term series to a (2, 1) Padé and then to find the root. The slightly different result is now

$$\alpha_3 = 3.590. \tag{15}$$

The natural continuation of this process is to calculate the (3, 1) Padé of the four-term series. The result is

$$\alpha_4 = 4.110. \tag{16}$$

(This is the only Padé that gives a stable positive root.)

Note that the sequence of roots α_2, α_3, α_4 is remarkably stable. It would be extremely interesting to calculate the roots of the (3, 2), (4, 2), (4, 3), ..., Padés.

We conclude this letter with a related observation. Recall that Casimir proposed a model for determining the charge of the electron. In this model the Coulomb repulsion of a compact charge distribution is balanced by an attractive zero-point energy [13]. Unfortunately, although the Casimir force for parallel plates is attractive, in a landmark paper Boyer showed that it is repulsive for a perfectly conducting spherical shell [14, 15], and thus no balance of forces is possible. However, with $\mathcal{PT}$-symmetric quantum electrodynamics such a balance is achievable.

In the absence of radiative corrections, the Casimir or zero-point energy of a perfectly conducting spherical shell of radius a is

$$E_{\text{Casimir}} = \frac{0.092\,35}{2a}\hbar c. \tag{17}$$

This energy results from fluctuations of the electromagnetic field inside and outside the shell. This result will be unchanged in the new theory if the boundary conditions are unaltered because the energy is independent of the coupling to the fermion. But now, rather than a Coulomb repulsion, we have an attraction because in effect $e \to \mathrm{i}e$. If a charge e is uniformly distributed over a spherical shell of radius a, that attractive energy is

$$E_{\text{Coulomb}} = -\frac{1}{8\pi}\frac{e^2}{a}. \tag{18}$$

Thus, stability is achieved if the two energies cancel:

$$E_{\text{Casimir}} + E_{\text{Coulomb}} = 0. \tag{19}$$

This implies a real, positive value for the fine structure constant:

$$\alpha = \frac{e^2}{4\pi\hbar c} = 0.092\,35. \tag{20}$$

This is an order of magnitude larger than the physical value $\frac{1}{137}$, and 40 times smaller than the value found above for a finite quantum electrodynamics. But what is significant here is that a positive solution for α actually exists.

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Appendix. Relation to Dyson's argument

Dyson's argument [16], made more precise in a series of papers by Itzykson *et al* and Bogomolny *et al* [17], seems to preclude the existence of a QED-like field theory for $e^2 < 0$. Dyson's arguments suggest that the perturbation series for quantum electrodynamics is probably divergent. This paper does not disagree with this conclusion. However, this conclusion is not relevant to the conjectures made in this paper, which we regard as plausible.

We emphasize that Dyson's argument is somewhat fuzzy. If we apply it to the anharmonic oscillator defined by the Hamiltonian $H = p^2 + x^2 + gx^4$, it goes as follows: for $g > 0$ there exists a vacuum state because the potential is bounded below. Now, replace g by $-g$. The potential in the resulting theory is unbounded below, so the vacuum is unstable in this new theory. Hence, the point $g = 0$ in the complex-g plane is a singularity of the ground state energy $E(g)$. We conclude that the perturbation series for $E(g)$, a series in powers of g, is divergent.

While this conclusion is correct, the Dyson argument leading to this conclusion is wrong! This is because it is not at all clear what is meant by replacing g by $-g$. One can obtain *many* totally different theories depending on *how* one replaces g by $-g$. For example, rotating g clockwise about $g = 0$ by 180° gives a complex value for $E(g)$. Rotating in the anticlockwise direction gives a *different* value for $E(g)$ (the complex conjugate value) even though the same Hamiltonian results. Most importantly: analytically continuing the Hamiltonian $H = p^2 + x^2 + gx^2(\mathrm{i}x)^a$ from $a = 0$ to $a = 2$ through positive values of the parameter a gives a value for the ground state energy that is *real* and *positive*. Indeed, the entire spectrum for the resulting Hamiltonian $H = p^2 + x^2 - gx^4 (g > 0)$ is *discrete*, *real*, and *positive*. (It is discrete, real, and positive for all $a \geqslant 0$.) This is the point of our work. See [1]. The idea of that work, and of the earlier [3], is that simply replacing the coupling constant by its negative (as in Dyson's argument) is ambiguous because it ignores the effect on the boundary conditions. A theory is not defined only by its Hamiltonian.

Here are some further examples. Consider the harmonic oscillator Hamiltonian $H = p^2+g^2x^2$. Replace g by $-g$. Although the Hamiltonian remains *invariant* under such a rotation, the spectrum changes sign! Also, consider the Hamiltonian $H = p^2 + g^2x^6 - 3gx^2 (g > 0)$. The ground state energy for this theory is exactly identically $E(g) = 0$. Now replace g by $-g$. The resulting Hamiltonian has a ground state that is strictly positive. Yet the analytic continuation of $E = 0$ is $E = 0$! The resolution of this paradox as explained in [3] is that merely replacing g by $-g$ following Dyson's argument is ambiguous. The boundary conditions must be analytically continued in g as well.

We are not sure that Dyson's large-order analysis of QED says anything about whether renormalized four-dimensional massless QED defined in the sense of [1] is stable or unstable. We do believe that such a theory is associated with the Feynman rules with e replaced by ie and that such a series is likely to be divergent. That is why the arguments given in the first part of our paper are heuristic and at most suggestive. However, these arguments are supported by our Casimir force calculation in the second part of our paper. We believe that our conclusions, while conjectural in nature, may well be valid.

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